JOURNAL OF COMPUTATIONAL PHYSICS 7, 172–177 (1971)

Quadratic Artificial Viscosity in Numerical Magnetic Gas Dynamics*

A form of artificial viscosity which is quadratic in the velocity gradient has been used extensively in numerical gas dynamics since its introduction by von Neumann and Richtmyer [1] twenty years ago. Its chief virtue is that it produces a shock transition whose width is constant (in Lagrangian coordinates) and independent of both the shock strength and the initial state of the medium. The purpose of this note is to investigate to what extent these desirable properties are preserved when a quadratic viscosity is used for magnetic gas dynamic shock problems.

We consider the case of a plane shock in an initially uniform, infinitely conducting fluid, together with an initially uniform magnetic field aligned perpendicular to the shock velocity. We follow the treatment given in Ref. [1], and analyze the structure of the steady-state shock by looking for running-wave solutions of the Lagrangian equations of motion. If the medium has initial density ρ_0 and satisfies an ideal gas equation of state, then the equations of motion may be written

$$\dot{V} = u_\alpha/\rho_0\,, \tag{1}$$

$$\rho_0\dot{u} = -(p + q + \mu H^2/2)_\alpha\,, \tag{2}$$

$$\dot{E} = -(p + q)\,\dot{V}, \tag{3}$$

$$E = pV/(\gamma - 1), \tag{4}$$

$$\dot{H}/H = -\dot{V}/V, \tag{5}$$

where

α = Lagrangian coordinate, taken as the initial position of the particle;
V = specific volume;
u = particle velocity;
E = internal energy/mass;
H = magnetic field;
p = fluid pressure;
q = viscosity term;
μ = permeability;
γ = ideal gas gamma.

* This work was supported by the Defense Atomic Support Agency.

Subscript α denotes partial differentiation with respect to α. Quantities in front of shock will bear the subscript 0; quantities behind shock will bear the subscript 1.

We assume a solution in which the dependent quantities are functions only of the variable $\omega = \alpha - st$, where s is the time rate of change of α at the shock front. (If u_0 is the initial particle velocity ahead of the shock and U is the Eulerian shock velocity, then $s = U - u_0$.) The equations of motion then reduce to a set of ordinary differential equations in the independent variable ω; these equations are easily solved, and in particular we may derive the following expression for q:

$$qV = \gamma C_1 V - \left(\frac{\gamma+1}{2}\right)(\rho_0 s V)^2 + \mu \frac{(\gamma-2)(H_0 V_0)^2}{2V} + C_2 . \tag{6}$$

The constants C_1, C_2 are evaluated by use of the boundary conditions $q = 0$ when $V = V_0$ and $V = V_1$, and the final result is

$$qV = \left(\frac{\gamma+1}{2}\right)\frac{(\rho_0 s)^2}{V}(V_0 - V)(V - V_1)(V - \xi), \tag{7}$$

where

$$\xi = \left(\frac{\gamma-2}{\gamma+1}\right)\frac{\mu H_1 H_0}{(\rho_0 s)^2} . \tag{8}$$

These results are independent of the form of q. We now specify that q is the von Neumann–Richtmyer quadratic viscosity, given by

$$q = \frac{l^2}{V}(u_\alpha)^2 \qquad \text{when} \qquad u_\alpha < 0, \tag{9}$$

and zero otherwise. (The constant l has the dimensions of length and is adjusted to give the desired shock sharpness in a numerical calculation.) This is equivalent to

$$q = \frac{(l\rho_0 s)^2}{V}\left(\frac{dV}{d\omega}\right)^2 \qquad \text{when} \qquad \frac{dV}{d\omega} > 0, \tag{10}$$

and zero otherwise. Inserting (10) into (7) we obtain the differential equation

$$d\omega = l\sqrt{\frac{2V}{(\gamma+1)(V_0 - V)(V - V_1)(V - \xi)}}\, dV, \tag{11}$$

from which the Lagrangian shock width follows by integration,

$$\Delta\alpha = l\sqrt{\frac{2}{\gamma+1}}\int_{V_1}^{V_0}\sqrt{\frac{V}{(V_0 - V)(V - V_1)(V - \xi)}}\, dV. \tag{12}$$

In general, this integral cannot be evaluated in closed form in terms of elementary functions; however, various special cases may be treated, and information gained about the behavior of $\Delta\alpha$. First of all, since $\xi = 0$ when $\gamma = 2$, we find that $\Delta\alpha$ is constant for this case, and independent of initial conditions; specifically, when $\gamma = 2$ we have

$$\Delta\alpha = l\sqrt{\frac{2}{3}}\int_{V_1}^{V_0}\frac{dV}{\sqrt{(V_0 - V)(V - V_1)}} = \pi l\sqrt{\frac{2}{3}}. \tag{13}$$

When $\gamma \neq 2$ it is useful to have ξ expressed in terms of ρ_1 and downstream conditions only; this may be achieved by using the magnetic gas dynamic Rankine–Hugoniot relations [2]. These equations relate conditions on both sides of the shock, and can be put in the form [3]

$$H_1/H_0 = \eta, \tag{14}$$

$$p_0\left\{\pi - 1 + \frac{\eta^2 - 1}{R_0}\right\} + \rho_0 s^2\left\{\frac{1}{\eta} - 1\right\} = 0, \tag{15}$$

$$p_0\left\{\frac{\gamma}{\gamma - 1}\left[\frac{\pi}{\eta} - 1\right] + \frac{2(\eta - 1)}{R_0}\right\} + \frac{1}{2}\rho_0 s^2\left\{\frac{1}{\eta^2} - 1\right\} = 0, \tag{16}$$

where

$$\pi = \frac{p_1}{p_0}, \qquad \eta = \frac{V_0}{V_1}, \qquad R_0 = \frac{2p_0}{\mu H_0^2}.$$

Equations (15) and (16) constitute a set of two linear equations in the two variables p_0 and $\rho_0 s^2$; a solution exists only if the determinant of the coefficients vanishes, which leads to the condition

$$\frac{\pi - 1}{\eta - 1} = \frac{2\gamma/(\gamma - 1) + (\eta - 1)^2/R_0}{(\gamma + 1)/(\gamma - 1) - \eta}. \tag{17}$$

Solving for $\rho_0 s^2$ yields

$$s^2 = \frac{p_0}{\rho_0}\eta\left[\frac{\pi - 1}{\eta - 1} + \frac{\eta + 1}{R_0}\right]. \tag{18}$$

Using (14), (17), and (18) in Eq. (8) then leads to

$$\xi = 2V_0\left(\frac{\gamma - 2}{\gamma + 1}\right)\left[\frac{\sigma - \eta}{(R_0 + 1)(\sigma + 1) + \eta(\sigma - 3)}\right], \tag{19}$$

where $\sigma = (\gamma + 1)/(\gamma - 1)$. The range of η is $1 < \eta < \sigma$, and at the extremities of this range we have

$$\xi = \left(\frac{\gamma - 2}{\gamma + 1}\right)\left(\frac{2V_0}{2 + \gamma R_0}\right) \quad \text{at} \quad \eta = 1 \quad \text{(weak shock)}, \tag{20}$$

$$\xi = 0 \quad \text{at} \quad \eta = \sigma \quad \text{(strong shock)}. \tag{21}$$

For $\gamma > 2$ we have $\xi > 0$, and it can be shown from (19) that $d\xi/d\eta < 0$. Therefore, using (20), we may show that

$$\xi \leqslant \left(\frac{\gamma - 2}{\gamma + 1}\right)\frac{2V_0}{(2 + \gamma R_0)} < \left(\frac{\gamma - 2}{\gamma + 1}\right) V_0 < \left(\frac{\gamma - 1}{\gamma + 1}\right) V_0 \leqslant V_1 . \tag{22}$$

Since $\xi < 0$ for $\gamma < 2$, we may then write the following inequality, valid for all γ:

$$\xi < V_1 . \tag{23}$$

This inequality insures that the integrand of (12) is always real, and since the integral converges we conclude that $\Delta\alpha$ is finite for all γ.

It follows from Eqs. (12, 21) that the shock width for an infinitely strong shock is given by

$$(\Delta\alpha)_s = \pi l \sqrt{\frac{2}{\gamma + 1}} \tag{24}$$

for any γ. To derive results for other shock strengths, it is useful to transform (12) to another form. It can be shown by contour integration that (12) is equivalent to

$$\Delta\alpha = (\Delta\alpha)_s + l \sqrt{\frac{2}{\gamma + 1}}\, I(\xi), \tag{25}$$

where

$$I(\xi) = \int_0^{\xi} \sqrt{\frac{x}{(V_0 - x)(V_1 - x)(\xi - x)}}\, dx$$

$$(-\infty < \xi < V_1). \tag{26}$$

For $\gamma > 2$ we have $\xi \geqslant 0$ and $I(\xi) \geqslant 0$; it therefore follows from (25) that $\Delta\alpha \geqslant (\Delta\alpha)_s$ when $\gamma > 2$. Similarly, when $\gamma < 2$ we have $\xi \leqslant 0$, $I(\xi) \leqslant 0$, and $\Delta\alpha \leqslant (\Delta\alpha)_s$ for this case.

For an infinitely weak shock $V_0 = V_1$,

$$I(\xi) = \int_0^\xi \frac{\sqrt{x}\, dx}{(V_0 - x)\sqrt{\xi - x}} = \pi \left(\sqrt{\frac{V_0}{V_0 - \xi}} - 1\right)$$
$$(-\infty < \xi < V_0). \tag{27}$$

Putting (27) into (25), and using the value of ξ given in (20), we obtain the shock width ratio,

$$\frac{(\Delta\alpha)_W}{(\Delta\alpha)_S} = \sqrt{\frac{\mathscr{L}}{2 - \gamma + \mathscr{L}}}, \tag{28}$$

where

$$\mathscr{L} = (\gamma + 1)\left(1 + \frac{\gamma}{2} R_0\right), \tag{29}$$

$$(\Delta\alpha)_W = \text{shock width for weak shock,}$$

$$(\Delta\alpha)_S = \text{shock width for strong shock.}$$

We may assume that the maximum variation in shock width occurs when an infinitely strong shock decays to an infinitely weak one. (To prove this rigorously, we would have to show that the integral in (12) is a monotonically increasing (decreasing) function of η for $\gamma < 2$ $(\gamma > 2)$ in the range $1 < \eta < \sigma$. This is somewhat troublesome to demonstrate analytically for general values of γ, V_0, and R_0; however, numerical calculations suggest that this is indeed the case.) Granted the assumption, it is then clear that the extreme values of $(\Delta\alpha)_W/(\Delta\alpha)_S$ yield absolute limits for the variation of shock width when a quadratic viscosity is used for the type of problem we have discussed.

For instance, when $\gamma < 2$ we find that $(\Delta\alpha)_W/(\Delta\alpha)_S$ takes its minimum value at $R_0 = 0$, $\gamma = 1$ (values of γ less than one are not physically admissable), and this minimum value is $\sqrt{\frac{2}{3}} = 0.8165$. Consequently, when $\gamma < 2$ we must always have

$$(\Delta\alpha)_S > (\Delta\alpha)_W > 0.8165(\Delta\alpha)_S.$$

This range of shock width variation is not likely to cause any difficulty in practical calculations.

Similarly, when $\gamma > 2$, we find that $(\Delta\alpha)_W/(\Delta\alpha)_S$ takes on a maximum with respect to R_0 when $R_0 = 0$, and the value of this maximum is $\sqrt{(\gamma + 1)/3}$. Consequently, when $\gamma > 2$ we must always have

$$(\Delta\alpha)_S < (\Delta\alpha)_W < (\Delta\alpha)_S \sqrt{\frac{\gamma + 1}{3}}.$$

For most physically reasonable values of gamma, this range of variation is probably acceptable in practical numerical calculations. Thus, for a gamma as high as eight, the shock width for a weak shock cannot be larger than 1.7321 times the shock width for a strong shock.

In conclusion, we point out that these results apply only to the simplest case of a plane, magnetic gas dynamic shock in which the magnetic field is perpendicular to the shock velocity. When the magnetic field is inclined to the direction of the shock velocity, the situation becomes more complex and the shock width may depend on the angle between the field and the velocity. Such problems were not considered here.

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RECEIVED: March 2, 1970

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